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$$= x\sqrt{(1-x^2)}\left(1+\frac{h}{x}-\frac{xh}{1-x^2}\right)+\frac{1}{x}\cdot\sqrt{(1-x^2)}\left(1-\frac{h}{2(1-x)}-\frac{h}{x}\right)$$

$$= u + h\left(\sqrt{(1-x^2)}-\frac{x^2}{\sqrt{(1-x^2)}}-\frac{1}{2x\sqrt{(1-x)}}-\frac{\sqrt{(1-x)}}{x^2}\right).$$

We must now put the coefficient of h equal to 0, and we shall have

$$\sqrt{(1-x^2)}-\frac{x}{\sqrt{(1-x^2)}}-\frac{1}{2x\sqrt{(1-x)}}-\frac{\sqrt{(1-x)}}{x^2}=0.$$

The solution of this equation will make known the required values of x Take the equation $u = xe^{-x}$, then

$$u' = (x+h)e^{-x+h} = (x+h)e^{-x}.e^{-h} = e^{-x}(x+h)(1-h+\tfrac{1}{2}h^2)$$

$$= e^{-x}(x-hx+\tfrac{1}{2}h^2x+h-h^2) = u+\psi'x.h+\tfrac{1}{2}\psi''x.h^2.$$

$$\therefore e^{-x}(1-x)=0,\ x=1.$$

Since $\psi''x = e^{-x}(\frac{1}{2}x-1)$, the value of $x = 1$, gives $\psi''x = e^{-1}(\frac{1}{2}-1) = -\frac{1}{2}e^{-1}$, a negative quantity, so that this value of x makes $u = e^{-1}$, a maximum.

We could easily extend this demonstration to the case of two independent variables.

ON THE SOLUTION OF CUBIC AND BIQUADRATIC EQUATIONS.

BY G. W. HILL.

In nearly all treatises on algebra the solution of these equations is presented as accomplished by the aid of analytical artifices, which one seems, by some happy hazard, to have stumbled upon. No doubt the processes were found in this manner by the original discoverers, Tartaglia, Carden and Ferrari. But, for many reasons, it would be better to treat the subject as one demanding invention rather than artifice. The equations can, as it were, be interrogated and compelled to yield up their secrets, if they have any.

To say that an equation is solvable algebraically, is to say that an algebraic expression can be found equivalent to the general root, that is, one involving a finite number of the operations of addition, subtraction, multiplication, division and the extraction of roots of prime degree. If the expression does not involve the last mentioned operation, it is called rational, and if free from the two last, integral.

However complex an algebraic expression involving radicals may be, it it is evident that there must be at least one radical which is involved in it rationally. Supposing this to be denoted by $R^{\frac{1}{n}}$, n being a prime integer, it is not difficult to convince one's self that, by the proper reductions, the expression can be exhibited thus

$$p_0 + p_1 R^{\frac{1}{n}} + p_2 R^{\frac{2}{n}} + \ . \ . \ . \ p_{n-1} R^{\frac{n-1}{n}} \ ,$$

where p_0, p_1, &c., do not involve the radical$R^{\frac{1}{n}}$. With no loss of generality we can suppose $p_1 = 1$; for if p_1 is not zero, we can multiply the quantity under the radical sign by p_1^n, and then take $(p_1^n R)^{\frac{1}{n}}$ as the radical; and in in the contrary case, if p_k is one of the quantities p which is not zero, the simplification can be accomplished by putting $R' = p_k^n R^k$. Then

$$p_0 + R^{\frac{1}{n}} + p_2 R^{\frac{2}{n}} + \ . \ . \ . + p_{n-1} R^{\frac{n-1}{n}}$$

may be regarded as the most general form of an algebraic expression.

Here may be enunciated a general proposition, which, although I am not aware that it has ever been proved, is doubtless true, and may be used for purposes of discovery. If an algebraic expression exists, equivalent to the general root of the equation

$$x^m + ax^{m-1} + bx^{m-2} + \ . \ . \ . + g = 0,$$

it can be exhibited in the above form, n being equal to one of the prime factors of m. Thus the algebraic expression of the root of the general equation of the 5th degree, if it existed, could be presented in the form

$$p_0 + R^{\frac{1}{5}} + p_2 R^{\frac{2}{5}} + p_3 R^{\frac{3}{5}} + p_4 R^{\frac{4}{5}} ,$$

and that of the 6th degree in either of the two forms

$$p_0 + R^{\frac{1}{3}} + p_2 R^{\frac{2}{3}} , \qquad p_0 + R^{\frac{1}{2}} .$$

Solution of Cubic Equations.

According to the foregoing proposition, the root of the general cubic equation

$$x^3 + ax^2 + bx + c = 0,$$

if it has an algebraic expression, must be presented in the form

$$x = p + R^{\frac{1}{3}} + p' R^{\frac{2}{3}} .$$

But since we suppose that this is an irreducible expression involving radicals, it follows that it must satisfy the given equation, whichever of its three values is attributed to the radical $\sqrt[3]{R}$. Thus calling either of the imaginary cube roots of unity a, the three roots of the cubic equation must be

$$x_1 = p + R^{\frac{1}{3}} + p'R^{\frac{2}{3}} ,$$

$$x_2 = p + a R^{\frac{1}{3}} + a^2 p'R^{\frac{2}{3}} ,$$

$$x_3 = p + a^2 R^{\frac{1}{3}} + a^4 p'R^{\frac{2}{3}} .$$

The first method that suggests itself for obtaining equations which shall give the values of p, p' and R, is to substitute these expressions in the symmetrical functions which are equivalent to the several coefficients a, b, c, viz.,

$$x_1 + x_2 + x_3 = -a , \quad x_1x_2 + x_1x_3 + x_2x_3 = b , \quad x_1x_2x_3 = c .$$

But a simpler proceeding is to employ the three symmetrical functions $\Sigma.x$, $\Sigma.x^2$ and $\Sigma.x^3$. Since any cube root, as $\sqrt[3]{R}$, is a root of $x^3 - R = 0$, in which the coefficients denoted above by a and b are each zero, it follows that the sum of the three cube roots of any quantity, as well as the sum of their squares, is zero. Now it is plain that if the value of x is raised to the n^{th} power,

$$x^n = A + BR^{\frac{1}{3}} + CR^{\frac{2}{3}} ,$$

where A, B and C are free from the radical $\sqrt[3]{R}$, and are consequently the same whichever of the three roots x denotes. Thus since $\Sigma.\sqrt[3]{R} = 0$, $\Sigma.\sqrt[3]{R^2} = 0$,

$$\Sigma.x^n = 3A .$$

Thus, for computing the value of $\Sigma.x^n$, we need only the part A which is free from the radical $\sqrt[3]{R}$. In this way we obtain and equate to their known values in terms of the coefficients a, b, c,

$$\Sigma.x = 3p = -a ,$$

$$\Sigma.x^2 = 3(p^2 + 2p'R) = a^2 - 2b ,$$

$$\Sigma.x^3 = 3(p^3 + R + 6pp'R + p'^3R^2) = -a^3 + 3ab - 3c .$$

These equations afford the values of p, p' and R; from the first two

$$p = -\frac{a}{3} , \qquad p'R = \frac{a^2 - 3b}{9} ,$$

and by substitution of these values in the last,

$$R^2 + \frac{2a^3 - 9ab + 27c}{27} R + \left(\frac{a^2\ 3+b}{9}\right)^3 = 0,$$

a quadratic equation in R; thus the general cubic admits of solution by radicals.

For the sake of brevity putting

$$A = \frac{a^2 - 3b}{9} \qquad B = -\frac{2a^3 - 9ab + 27c}{54},$$

we have

$$R = B \pm \sqrt{(B^2 - A^2)},$$

and as we may take at our option either of the two roots, we have choice of the two expressions for x,

$$x = -\tfrac{1}{3}a + [B + \sqrt{(B^2 - A^3)}]^{\frac{1}{3}} + A[B + \sqrt{(B^2 - A^3)}]^{-\frac{1}{3}}$$

$$x = -\tfrac{1}{3}a + [B - \sqrt{(B^2 - A^3)}]^{\frac{1}{3}} + A[B - \sqrt{(B^2 - A^2)}]^{-\frac{1}{3}}$$

The three values of x are obtained by attributing in succession to the single cube root appearing in either of these expressions its three values.

I do not know why almost all algebraists prefer to put the root in the form

$$x = -\tfrac{1}{3}a + \sqrt[3]{[B + \sqrt{(B^2 - A^3)}]} + \sqrt[3]{[B - \sqrt{(B^2 - A^3)}]}.$$

It is certainly easier in practice to make a division than an extraction of a cube root; moreover we are troubled, in the last form, with the selection of the proper three values out of the nine of which it is susceptible, a difficulty which does not occur in the two former expressions.

SOLUTION OF BIQUADRATIC EQUATIONS.

An algebraic expression for the root of the general equation of the fourth degree,

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

if it exists, can be presented in the form $P + \sqrt{Q}$. And if this denotes one of the roots, another will be $P - \sqrt{Q}$; but since x has four values, it is plain that P and Q must receive each two values. This condition will be fulfiled if we suppose that these quantities, in their turn, similarly to x, are rational functions of a second radical $\sqrt{R}$. Thus we put

$$P = p + \sqrt{R},$$
$$Q = q + q'\sqrt{R}.$$

Then we have

$$x = p + \sqrt{R} + \sqrt{(q + q'\sqrt{R})}.$$

The four values of x are obtained by giving in succession to the radicals $\sqrt{Q}$ and $\sqrt{R}$ all the values they are, in combination, susceptible of. Thus

$$x_1 = p + \sqrt{R} + \sqrt{(q + q'\sqrt{R})},$$
$$x_2 = p - \sqrt{R} + \sqrt{(q - q'\sqrt{R})},$$
$$x_3 = p + \sqrt{R} - \sqrt{(q + q'\sqrt{R})},$$
$$x_4 = p - \sqrt{R} - \sqrt{(q - q'\sqrt{R})}.$$

By aubstituting these in the four symmetrical functions $\Sigma.x$, $\Sigma.x^2$, $\Sigma.x^3$ and $\Sigma.x^4$, equations will be found determining p, q, q' R. Here again, in computing $\Sigma.x^n$ the radicals all disappear; for whenever a radical is present with one sign in any root, there is always another root in which it is present with the opposite sign; thus these *expressions* in pairs cancel each other. Then' in deriving $\Sigma.x^n$, it is only necessary to preserve the terms which are free from radicals. In this way we get

$$\begin{aligned}
\Sigma.x &= 4p && = -a\,,\\
\Sigma.x^2 &= 4[p^2+q+\mathrm{R}] && = +a^2-2b\,,\\
\Sigma.x^3 &= 4[p^3+3p(q+\mathrm{R})+3q'\mathrm{R}] && = -a^3+3ab-3c\,,\\
\Sigma.x^4 &= 4[(p^2+q+\mathrm{R})^2+(4p^2+12pq'+q'^2)\mathrm{R}+4q(p^2+\mathrm{R})]\\
&= a^4-4a^2b+4ac+2b^2-4d\,.
\end{aligned}$$

From which we derive

$$p=-\frac{a}{4}\,,\quad q+\mathrm{R}=\frac{3a^2-8b}{16}\,,\quad q'\mathrm{R}=-\frac{a^3-4ab+8c}{32}\,.$$

$$\mathrm{R}^3-\frac{3a^2-8b}{16}R^2+\frac{3a^4-16a^2b+16ac+16b^2-64d}{256}\mathrm{R}-\left(\frac{3a^3-4ab+8c}{64}\right)^2=0\,.$$

The last is a cubic equation in R, which, by the foregoing, is solvable by radicals; hence the general equation of the fourth degree is so solvable.

In forming the value of x we may attribute to R either of the three roots of this equation. When $a=0$, the case usually treated, the equations are simpler, viz.,

$$p=0\,,\qquad q+\mathrm{R}=-\tfrac{1}{2}b\,,\qquad q'\mathrm{R}=-\tfrac{1}{4}c\,,$$

$$\mathrm{R}^3+\frac{b}{2}R^2+\frac{b^2-4d}{16}\mathrm{R}-\frac{c^2}{64}=0\,.$$

If we should attempt to treat the general equation of the 5th degree in the preceding manner, we would be led to equations of higher degrees than the 5th, which must be regarded as a strong probable argument for the non-existence of an algebraic expression equivalent to the root of the general equation of this degree.

ADDITIONAL FORMULAE IN FINITE DIFFERENCES.

BY G. W. HILL.

Tne formulae I have given, (p. 144 of this Journal), give the values of the integrals $\int y dx$ and $\int\int dx^2$ for the series of values of x, .. $a-h$, a, $a+h$, ..